REAL INVERSION AND JUMP FORMULAE FOR THE LAPLACE TRANSFORM. PART I.

BY

Z. DITZIAN[1] AND A. JAKIMOVSKI[2]

ABSTRACT

Generalizations of the Laplace asymptotic method are obtained and real inversion formulae of the Post-Widder type for the Laplace transform are generalized.

§1. **Introduction.** In §2 of this paper we shall obtain generalizations of what are known as real inversion and jump formulae of the Post-Widder type for the Laplace transform. From our generalized inversion formula and certain heuristic considerations we shall obtain some new jump formulae.

Generalizations of the Laplace asymptotic method necessary for proving the results of §2 are stated and proved in §3.

§2. **Generalization of Post-Widder inversion and jump formula.** Here the Laplace transform $f(s)$ of the function $\phi(t)$ is defined by

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt \equiv \lim_{R\to\infty}\int_0^R e^{-st}\phi(t)dt \tag{1.2}$$

where $\phi(t) \in L_1(0,R)$ for each $R > 0$ and the right hand side is supposed convergent for some finite complex s.

The Laplace-Stieltjes transform of the function $\alpha(t)$ is defined by

$$f(s) = \int_0^\infty e^{-st}d\alpha(t) \equiv \lim_{R\to\infty}\int_0^R e^{-st}d\alpha(t) \tag{2.2}$$

where $\alpha(t)$ is of bounded variation in the interval $[0,R]$ for each $R > 0$, $\alpha(0) = 0$, $\alpha(t) = 1/2(\alpha(t+) + \alpha(t-))$ and the right hand side of (2.2) is supposed convergent for some finite complex s.

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Here $\phi(x \pm)$ are defined by

$$\lim_{h \to 0} \phi(x \pm h) \equiv \phi(x \pm) \tag{3.2}$$

if these limits exist.
$\phi(x \pm 0)$ will denote the numbers for which

$$\int_0^h [\phi(x \pm y) - \phi(x \pm 0)]\, dy = o(h) \text{ as } h \downarrow 0 \tag{4.2}$$

if such numbers exist. If $\phi(x+0) = \phi(x-0)$ then x is a Lebesgue point of $\phi(u)$.

Post [5] obtained a real inversion formula. Widder [6] and also Feller [2], Dubourdieu [1], and especially Pollard ([3] and [4]) generalized Post's result. The final result obtained by Pollard is the following (see [5], Theorem 1.1, second part):

THEOREM A. *Suppose $f(s)$ is a Laplace transform and both $\phi(t \pm 0)$ exist; let $\{a_k\}$, $k \geqq 1$, be any sequence satisfying $a_k = o(k^{1/2})$, $k \to \infty$; then*

$$\lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k + a_k}{t} \right)^{k+1} f^{(k)}\left(\frac{k + a_k}{t} \right) = \frac{1}{2}[\phi(t+0) + \phi(t-0)] \tag{5.2}$$

Pollard ([5], Theorem 1.1) stated that if in addition to the hypotheses of Theorem A we assume $\phi(t+0) = \phi(t-0)$, then (5.2) is true for any sequence $\{a_k\}$ satisfying $a_k = o(k)$, $k \to \infty$. We could not prove this last result and shall show by an example that one of the main steps in his proof is incorrect.

The step in question (see [5] top of page 449) is that, for $0 < \delta < 1$ and $\theta_k = o(k)$, $k \to \infty$,

$$\left| \int_{k/(k+\theta_k)}^1 P_k(u)du \right| \leqq \int_0^{1-\delta} P_k(u)du \tag{5a.2}$$

and

$$\left| \int_{k/(k+\theta_k)}^1 P_k(u)du \right| \leqq \int_{1+\delta}^{\infty} P_k(u)du$$

where

$$P_k(u) = \frac{(k+\theta_k)^{k+1}}{(k-1)!} e^{-(k+\theta_k)u} u^{k-1}(1-u)\left(1 - \frac{k+\theta_k}{k} u\right).$$

We shall show that these two inequalities are not true if, for example, $\sqrt{k} = o(\theta_k)$, $k \to \infty$.

For $k \geqq k_0$ (since $\theta_k = o(k)$) we have

$$\begin{aligned} 0 < (1-u)\left(1 - \frac{k+\theta_k}{k} u\right) &= 1 - \left(1 + \frac{k+\theta_k}{k}\right) u + \frac{k+\theta_k}{k} u^2 \\ &\leqq 1 + 3u + 3u^2 \\ &\leqq 6, \text{ for } 0 \leqq u \leqq 1 - \delta < 1. \end{aligned}$$

Hence, for $k \geqq k_0$,

$$\int_0^{1-\delta} P_k(u)du \leqq 6\,\frac{(k+\theta_k)^{k+1}}{(k-1)!}\int_0^{1-\delta} e^{-(k+\theta_k)u}u^{k-1}\,du$$

The integrand in the last integral is increasing from $u = 0$ to

$$u = \frac{k-1}{k+\theta_k}(\to 1 \text{ as } k \to \infty).$$

Therefore for $k > k_1 \geqq k_0$

$$\int_0^{1-\delta} P_k(u)du \leqq 6(1-\delta)^k\,\frac{(k+\theta_k)^{k+1}}{(k-1)!}\,e^{-(1-\delta)(k+\theta_k)}$$

(and by Stirling's formula)

$$\sim A\exp\left\{-k\log\frac{1}{1-\delta} + (k+1)\log(k+\theta_k) - (1-\delta)(k+\theta_k) - \left(k-\frac{1}{2}\right)\log k + k\right\}$$

$$= A\exp\left\{-k\left[\log\frac{1}{1-\delta} - \delta + o(1)\right] + \frac{3}{2}\log k\right\}$$

(and since $\delta < \log 1/(1-\delta)$ for $0 < \delta < 1$)

$$\to 0 \text{ as } k \to \infty.$$

Hence

(5b.2)
$$\lim_{k\to\infty}\int_0^{1-\delta} P_k(u)du = 0$$

Now

$$\frac{d}{du}\{e^{-(k+\theta_k)u}u^k\} = ke^{-(k+\theta_k)u}u^{k-1}\left(1 - \frac{k+\theta_k}{k}u\right).$$

Hence

(5c.2)
$$\int_{k/(k+\theta_k)}^{1} P_k(u)du = \frac{(k+\theta_k)^{k+1}}{k!}\int_{k/(k+\theta_k)}^{1}(1-u)\,\frac{d}{du}\{u^k e^{-(k+\theta_k)u}\}\,du$$

$$= -\frac{k^k e^{-k}}{k!}\theta_k + \frac{(k+\theta_k)^{k+1}}{k!}\int_{k/(k+\theta_k)}^{1} u^k e^{-(k+\theta_k)u}\,du$$

$$= -\frac{[1+o(1)]\theta_k}{\sqrt{2\pi k}} + \frac{1}{k!}\int_k^{k+\theta_k} v^k e^{-v}\,dv.$$

We have

$$\left|\frac{1}{k!}\int_k^{k+\theta_k} v^k e^{-v} dv\right| \leqq \frac{1}{k!}\int_0^\infty v^k e^{-v} dv = 1.$$

If now $\sqrt{k} = o(\theta_k)$ then $|\theta_k/\sqrt{k}| \to \infty$ as $k \to \infty$ and

$$\lim_{k\to\infty}\left|\int_{k/(k+\theta_k)}^1 P_k(u)du\right| = \infty. \tag{5d.2}$$

Relations (5b.2) and (5d.2) are in contradiction with (5a.2). Similarly it can be shown that the second inequality used by Pollard is not true if $\sqrt{k} = o(\theta_k), k \to \infty$.

Our first result is a generalization of Theorem A.

In order to state our theorem we need the following notation.

a) A sequence $\{a_k\}$, $k \geqq 1$, belongs to class $A(\lambda)$ for some real λ if

$$a_k - \lambda\sqrt{k} = o(\sqrt{k}), k \to \infty.$$

b) A sequence $\{a_k\}$ belongs to class B if $a_k = o(k)$, $k \to \infty$.

c) A sequence $\{a_k\}$ belongs respectively to class $B^+(B^-)$ if $\{a_k\} \in B$, $|a_k|k^{-1/2} \to \infty$ as $k \to \infty$ and for some $k \geqq k_0$, $a_k > 0 (a_k < 0)$.

d) A sequence $\{a_k\}$ belongs to class A^* if $a_k = O(\sqrt{k})$ as $k \to \infty$.

Denote by $N(\lambda)$ the normal distribution function,

$$N(\lambda) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\lambda} e^{-u^2/2} du. \tag{6.2}$$

For $k = 1, 2, \ldots$, $t > 0$ and a given sequence $\{a_k\}$ the operator $L_{k,t,a_k}[f(x)]$ is defined by

$$L_{k,t,a_k}[f(s)] \equiv \frac{(-1)^k}{k!}\left(\frac{k+a_k}{t}\right)^{k+1} f^{(k)}\left(\frac{k+a_k}{t}\right). \tag{7.2}$$

THEOREM 1.2. *Suppose $f(s)$ is the Laplace transform of $\phi(u)$. For fixed $t > 0$,*

(i) *If $\{\alpha_k\} \in A(\lambda)$ and both $\phi(t \pm 0)$ exist then*

$$\lim_{k\to\infty} L_{k,t,a_k}[f(x)] = N(\lambda)\phi(t-0) + (1 - N(\lambda))\phi(t+0).$$

(ii) *If $\{a_k\} \in B^+$ and $\phi(t-)$ exists then*

$$\lim_{k\to\infty} L_{k,t,a_k}[f(x)] = \phi(t-).$$

(iii) *If $\{a_k\} \in B^-$ and $\phi(t+)$ exists then*

$$\lim_{k\to\infty} L_{k,t,a_k}[f(x)] = \phi(t+).$$

(iv) *If* $\{a_k\} \in B$ *and* $\phi(u)$ *is continuous at a point* $t > 0$ *then*

$$\lim_{k\to\infty} L_{k,t,a_k}[f(x)] = \phi(t).$$

(v) *If* $\{a_k\} \in A^*$ *and both* $\phi(t \pm 0)$ *exist and* $\phi(t+0) = \phi(t-0)$, *then*

$$\lim_{k\to\infty} L_{k,t,a_k}[f(x)] = \phi(t+0).$$

We shall obtain the following two analogous results for the Laplace-Stieltjes transform. We begin by defining the operator S_{k,t,a_k}:

Given $t > 0$ and a sequence $\{a_k\}$, $k = 1,2,\dots$, we define

$$S_{k,t,a_k}[f(s)] \equiv f(\infty) + (-1)^{k+1} \int_{(k+a_k)/t}^{\infty} f^{(k+1)}(u) \frac{u^k}{k!} du \tag{8.2}$$

THEOREM 2.2. *Suppose* $f(s)$ *is the Laplace-Stieltjes transform of* $\alpha(t)$. *Then*

$$\lim_{k\to\infty} S_{k,t,a_k}[f(s)] = \begin{cases} N(\lambda)\alpha(t-) + (1-N(\lambda))\alpha(t+) & \text{if } \{a_k\} \in A(\lambda) \\ \alpha(t-) & \text{if } \{a_k\} \in B^+ \\ \alpha(t+) & \text{if } \{a_k\} \in B^- \\ \alpha(t) & \text{if } \alpha(t-) = \alpha(t+) \text{ and } \{a_k\} \in B. \end{cases}$$

THEOREM 3.2. *If* $f(s)$ *is the Laplace-Stieltjes transform of* $\alpha(u)$, *then*

$$\lim_{k\to\infty} \int_0^t L_{k,u,a_k}[f(s)]du = \begin{cases} N(\lambda)\alpha(t-) + (1-N(\lambda))\alpha(t+) - \alpha(0+) & \text{if } \{a_k\} \in A(\lambda) \\ \alpha(t-) - \alpha(0+) & \text{if } \{a_k\} \in B^+ \\ \alpha(t+) - \alpha(0+) & \text{if } \{a_k\} \in B^- \\ \alpha(t) - \alpha(0+) & \text{if } \alpha(t-) = \alpha(t+) \text{ and } \{a_k\} \in B \end{cases}$$

If we choose in the last three theorems $\alpha_k = o(k^{1/2})$ $k \to \infty$ we get Pollard's results [5].

Theorems 1.2, 2.2 and 3.2 will be proved in §4 by using improvements of the Laplace asymptotic method which are stated and proved in §3. These improvements are stated in a slightly more general form than is needed for proving the results of this paper. The more general form will be needed in a later paper.

From Theorems 1.2, 2.2 and 3.2 we can deduce immediately the following trivial jump formulae.

COROLLARY 1.2. *Suppose* $f(s)$ *is the Laplace transform of* $\phi(u)$. *If for some* $t > 0$ *both* $\phi(t \pm 0)$ *exist and* $\{a_k\} \in A(\lambda_1)$, $\{b_k\} \in A(\lambda_2)$ $(\lambda_1 \neq \lambda_2)$, *then*

$$\frac{1}{N(\lambda_1) - N(\lambda_2)} \lim_{k\to\infty} \{L_{k,t,b_k}[f(s)] - L_{k,t,a_k}[f(s)]\} = \phi(t+0) - \phi(t-0).$$

COROLLARY 2.2. *Suppose $f(s)$ is the Laplace transform of $\phi(u)$. If for some $t > 0$ both $\phi(t \pm)$ exist and $\{a_k\} \in B^+$, $\{b_k\} \in B^-$*, then

$$\lim_{k\to\infty} \{L_{k,t,b_k}[f(s)] - L_{k,t\,a_k}[f(s)]\} = \phi(t+) - \phi(t-).$$

COROLLARY 3.2. *Suppose $f(s)$ is the Laplace-Stieltjes transform of $\alpha(u)$. For fixed $t > 0$ we have*

(i) *If $\{a_k\} \in A(\lambda_1)$ and $\{b_k\} \in A(\lambda_2)$ $(\lambda_1 \neq \lambda_2)$, then*

$$\text{(a)} \qquad \frac{1}{N(\lambda_1) - N(\lambda_2)} \lim_{k\to\infty} (-1)^{k+1} \int_{(k+b_k)/t}^{(k+a_k)/t} \frac{u^k}{k!} f^{(k+1)}(u)du = \alpha(t+) - \alpha(t-)$$

and

$$\text{(b)} \qquad \frac{1}{N(\lambda_1) - N(\lambda_2)} \lim_{k\to\infty} \int_0^t \{L_{k,u,b_k}[f(x)] - L_{k,u,a_k}[f(x)]\}du$$

$$= \alpha(t+) - \alpha(t-).$$

(ii) *If $\{a_k\} \in B^+$ and $\{b_k\} \in B^-$, then*

$$\text{(c)} \quad \lim_{k\to\infty} \{S_{k,t,b_k}[f(x)] - S_{k,t,a_k}[f(x)]\} = \alpha(t+) - \alpha(t-)$$

and

$$\text{(d)} \quad \lim_{k\to\infty} \int_0^t \{L_{k,u,b_k}[f(x)] - L_{k,u,a_k}[f(x)]\}du = \alpha(t+) - \alpha(t-).$$

The following heuristic considerations lead to a new non-trivial jump formula. Choose in Corollary 1.2, $b_k - a_k = (\lambda_2 - \lambda_1)\sqrt{k}$

$$\phi(t+0) - \phi(t-0) = \lim_{\lambda_1\to\lambda_2} \frac{1}{N(\lambda_1) - N(\lambda_2)} \lim_{k\to\infty} \{L_{k,t,b_k}[f(x)] - L_{k,t,a_k}[f(x)]\}$$

(changing formally the order of the limits)

$$= \lim_{k\to\infty} \lim_{\lambda_1\to\lambda_2} \frac{\lambda_1 - \lambda_2}{N(\lambda_1) - N(\lambda_2)} \cdot \frac{1}{\lambda_1 - \lambda_2} \cdot \frac{(-1)^k}{k!} \cdot \left\{ \left(\frac{k+b_k}{t}\right)^{k+1} f^{(k)}\left(\frac{k+b_k}{t}\right) - \left(\frac{k+a_k}{t}\right)^{k+1} f^{(k)}\left(\frac{k+a_k}{t}\right) \right\} =$$

$$= \lim_{k\to\infty} \sqrt{2\pi k}\, e^{\lambda^2/2} \lim_{\lambda_1\to\lambda_2} \frac{1}{(\lambda_1 - \lambda_2)\sqrt{k}} \frac{(-1)^k}{k!} \left\{ \left(\frac{k+b_k}{t}\right)^{k+1} f^{(k)}\left(\frac{k+b_k}{t}\right) - \left(\frac{k+b_k+(\lambda_1-\lambda_2)\sqrt{k}}{t}\right)^{k+1} f^{(k)}\left(\frac{k+b_k+(\lambda_1-\lambda_2)\sqrt{k}}{t}\right) \right\}$$

$$= \lim_{k\to\infty} \sqrt{2\pi k}\,\phi^{\lambda_2 2/2}\frac{(-1)^{k+1}}{k!}\left\{\left(\frac{k+b_k}{t}\right)^k\left(\frac{k+1}{t}\right) f^{(k)}\left(\frac{k+b_k}{t}\right) + \frac{1}{t}\left(\frac{k+b_k}{t}\right)^{k+1} f^{(k+1)}\left(\frac{k+b_k}{t}\right)\right\}.$$

This suggests Theorem 4.2, which we shall prove formally below in §4.

THEOREM 4.2. *Suppose $f(s)$ is the Laplace transform of $\phi(u)$ and that for some $t>0$, both $\phi(t\pm 0)$ exist. Let $\{a_k\}\in A(\lambda)$, and $\theta_k = o(\sqrt{k})$, $k\to\infty$. Then*

$$\lim_{k\to\infty}\sqrt{2\pi k}\,\frac{(-1)^k}{k!}\left\{\left(\frac{k+a_k}{t}\right)^k\frac{k+\theta_k}{t} f^{(k)}\left(\frac{k+a_k}{t}\right) + \frac{1}{t}\left(\frac{k+a_k}{t}\right)^{k+1} f^{(k+1)}\left(\frac{k+a_k}{t}\right)\right\} = \phi(t+0-\phi(t-0).$$

Choosingin Theorem 4.2 $\lambda = 0$, $a_k = 0$ and $\theta_k = 0$ for $k \geqq 1$ we get

COROLLARY 4.2. *Suppose $f(s)$ is the Laplace transform of $\phi(u)$. If for some $t>0$ both $\phi(t\pm 0)$ exist, then*

$$\lim_{k\to\infty}\frac{\sqrt{2\pi k}}{k!}\left(-\frac{k}{t}\right)^{k+1}\left[f^{(k)}\left(\frac{k}{t}\right)+\frac{1}{t}f^{(k+1)}\left(\frac{k}{t}\right)\right] = \phi(t+0)-\phi(t-0).$$

A formal computation similar to that used in obtaining Theorem 4.2 yields

THEOREM 5.2. *Suppose $f(s)$ is the Laplace-Stieltjes transform of $\alpha(t)$ and let $\{a_k\}\in A(\lambda)$. Then*

$$\lim_{k\to\infty}(-1)^{k+1} e^{\lambda^2/2}\frac{\sqrt{2\pi k}}{k!}\left(\frac{k+a_k}{t}\right)^k f^{(k)}\left(\frac{k+a_k}{t}\right) = \alpha(t+)-\alpha(t-)$$

Choosing in Theorem 5.2 $\lambda = 0$, $a_k = 0$ for $k \geqq 1$ we obtain the Widder jump formula (see [7] p. 298).

In Theorems 4.2 and 5.2 $\{a_k\}\in A(\lambda)$, therefore $a_k = O(\sqrt{k})$ $(k\to\infty)$. Hence

$$\left(\frac{k+a_k}{k}\right)^k = e^{k\log(1+(a_k/k))} \sim e^{a_k}e^{-1/2(a_k^2/k)} \sim e^{a_k-(\lambda^2/2)}(k\to\infty).$$

Therefore we may state the results of Theorem 4.2 and of Theorem 5.2 in, respectively, the following two equivalent forms:

$$\lim_{k\to\infty}\left(-\frac{1}{t}\right)^{k+1} e^{k+a_k}\left\{(k+\theta_k)f^{(k)}\left(\frac{k+a_k}{t}\right)+\frac{k+a_k}{t}f^{(k+1)}\left(\frac{k+a_k}{t}\right)\right\} = \phi(t+0)-\phi(t-0)$$

$$\lim_{k\to\infty} e^{k+a_k}\left(-\frac{1}{t}\right)^k f^{(k)}\left(\frac{k+a_k}{t}\right) = \alpha(t+)-\alpha(t-).$$

§3. **The Laplace asymptotic method.** In this section we shall obtain and prove some results which are the basis for the proof of our theorems in §2 and in part 2 of this paper. The results of this section are generalizations of the classic Laplace asymptotic method.

THEOREM 1.3. *Let* (i) a, b, η, δ *and* λ *be real numbers satisfying* $a < b$, $\delta > 0$ *and* $0 < \eta < b - a$.

(ii) $h(x) \in C^2(a - \delta \leqq x \leqq a + \eta)$, $h'(a) = 0$, $h''(a) < 0$ *and* $h(x)$ *is non increasing in* $a \leqq x \leqq b$.

(iii) $\{a_k\} \in A(\lambda)$.

(iv) $\{g(k)\}$, $k \geqq 1$, *is a sequence of real numbers satisfying* $g(k) \sim k, k \to \infty$. *Then*

$$(1.3)\quad \lim_{k \to \infty} \sqrt{\frac{k}{2\pi}}\, e^{-g(k)h(a)} \sqrt{-h''(a)} \int_{a + a_k \cdot g(k)^{-1}}^{b} e^{g(k)h(x)}\, dx = 1 - N(\lambda \sqrt{-h''(a)})$$

Proof.

$$\sqrt{\frac{-kh''(a)}{2\pi}} \int_{a+a_k \cdot g(k)^{-1}}^{b} \exp[g(k)(h(x) - h(a))]dx$$

$$= \sqrt{\frac{-kh''(a)}{2\pi}} \left\{ \int_{a+a_k \cdot g(k)^{-1}}^{a+\eta_1} + \int_{a+\eta_1}^{b} \right\} (\quad)dx \equiv I_k + J_k \text{ (say).}$$

By the arguments used in Widder [7] p. 278 we see that, for any $0 < \eta_1 < \eta$,

$$(2.3)\qquad \lim_{k \to \infty} J_k = 0$$

By (ii), for $x \in [a - \delta, a + \eta_1]$, we have $h(x) - h(a) = [(x-a)^2/2]\, h''(\xi(x))$ where $\xi(x) \in [x, a]$ if $x < a$ or $\xi(x) \in [a, x]$ if $x > a$. Now

$$I_k = \sqrt{\frac{-kh''(a)}{2\pi}} \int_{a+a_k \cdot g(k)^{-1}}^{a+\eta_1} \exp\left[g(k)\, h''(\xi(x)) \frac{(x-a)^2}{2} \right] dx.$$

For any fixed ε satisfying $-h''(a) > \varepsilon > 0$ we can define η_1, $0 < \eta_1 < \eta$, and δ_1, $0 < \delta_1 < \delta$ so that, for $x \in [a - \delta_1, a + \eta_1]$,

$$(3.3)\qquad h''(a) - \varepsilon \leqq h''(x) \leqq h''(a) + \varepsilon.$$

Denote, respectively,

$$I_k^{\pm} \equiv \sqrt{\frac{-kh''(a)}{2\pi}} \int_{a+a_k \cdot g(k)^{-1}}^{a+\eta_1} \exp\left[g(k)(h''(a) \pm \varepsilon)\, \frac{(x-a)^2}{2} \right] dx$$

$$= \sqrt{\frac{k}{g(k)}} \sqrt{\frac{h''(a)}{h''(a) \pm \varepsilon}} \big(N(\eta_1 g(k)^{1/2} \sqrt{-(h''(a) \pm \varepsilon)})$$

$$- N(a_k g(k)^{-1/2} \sqrt{-(h''(a) \mp \varepsilon)})\big)$$

$N(u)$ is a continuous function, and hence

$$\lim_{k\to\infty} I_k^{\pm} = \sqrt{\frac{h''(a)}{h''(a)\pm\varepsilon}}\ (1 - N(\lambda\ \sqrt{-(h''(a)\pm\varepsilon)})). \tag{4.3}$$

By (2.3) and the definitions of $I_k^{\pm}$and I_k, $I_k^- \leqq I_k \leqq I_k^+$. Hence by (4.3) and since $\varepsilon > 0$ is arbitrary we get

$$\lim_{k\to\infty} I_k = 1 - N(\lambda\sqrt{-h''(a)}) \tag{5.3}$$

Our result follows from (2.3) and (5.3) combined. Q.E.D.

COROLLARY 1.3. *Let* (i) a, b, η, δ *and* λ *be real numbers satisfying* $a < b$, $\delta > 0$ *and* $0 < \eta < b - a$.

(ii) $h(x) \in C^2(a - \eta \leqq x \leqq a + \delta)$, $h'(a) = 0$, $h''(a) < 0$ *and* $h(x)$ *is nondecreasing in* $b \leqq x \leqq a$.

(iii) $\{a_k\}$ and $\{g(k)\}$ *satisfy conditions* (iii) *and* (iv) *of Theorem* 1.3.
Then

$$\lim_{k\to\infty}\sqrt{\frac{-kh''(a)}{2\pi}}\ e^{-g(k)h(a)}\int_b^{a+a_k\cdot g(k)^{-1}} e^{g(k)h(x)}dx = N(\lambda\sqrt{-h''(a)}) \tag{6.3}$$

Proof. The proof follows from Theorem 1.3 by substituting $x = -z$.

THEOREM 2.3. *Suppose* (i) a, b, η, δ, λ, $h(x)$, $\{a_k\}$ *and* $\{g(k)\}$ *satisfy conditions* (i), (ii), (iii) *and* (iv) *of Theorem* 1.3.

(ii) *For the functions* $\phi_k(u)$, $k)\geqq)1$, *defined on* $[a + a_k g(k)^{-1}, b]$ *and for some finite* M *we have*

$$\int_{a+a_k\cdot g(k)^{-1}}^{b} |\phi_k(u)|\,du \leqq M \text{ for } k \geqq 1$$

(iii) $\lim_{k\to\infty}\phi_k(a + a_k . g(k)^{-1} + 0)$ *exists and is equal to* A.

(iv) *Denote*

$$\alpha_k(x) \equiv \int_{a+a_k\cdot g(k)^{-1}}^{x} [\phi_k(u) - \phi_k(a + a_k g(k)^{-1} + 0)]\,du$$

for $k \geqq 1$ *and* $x \in [a + a_k\cdot g(k)^{-1}, b]$. *Suppose that for each* $\varepsilon_1 > 0$ *there exists a sufficiently small* $\rho(\varepsilon_1)$, $0 < \rho(\varepsilon_1) < (b - a)/2$, *such that for each* $k \geqq 1$ *and all* x *satisfying* $0 \leqq x - (a + a_k\cdot g(k_{-6}) \leqq \rho(\varepsilon_1)$ *we have*

$$|\alpha_k(x)| \leqq \varepsilon_1 |x - (a + a_k g(k)^{-1})|.$$

Then

$$\lim_{k\to\infty}\sqrt{\frac{-kh''(a)}{2\pi}}\ e^{-g(k)h(a)}\int_{a+a_k\cdot g(k)^{-1}}^{b} \phi_k(x)e^{g(k)h(x)}\,dx \tag{7.3}$$
$$= A(1 - N(\lambda\sqrt{-h''(a)})).$$

Proof. In order to prove our theorem it is enough by (1.3) to show that

$$(8.3)\quad \lim_{k\to\infty} \sqrt{k} \int_{a+a_k\cdot g(k)^{-1}}^{b} [\phi_k(x) - A]\exp[g(k)(h(x)-h(a))]\,dx = 0.$$

(1.3) yields by condition (iii) that

$$(9.3)\quad \lim_{k\to\infty} \sqrt{k} \int_{a+a_k\cdot g(k)^{-1}}^{b} [\phi_k(a+a_k\cdot g(k)^{-1}+0) - A]\cdot \exp[g(k)(h(x)-h(a))]\,dx = 0.$$

In order to prove (8.3) it is enough by (9.3) to show that

$$I_k \equiv \sqrt{k} \int_{a+a_k\cdot g(k)^{-1}}^{b} [\phi_k(x) - \phi_k(a+a_k\cdot g(k)^{-1}+0)]\cdot \exp[g(k)(h(x)-h(a))]\,dx = o(1) \text{ as } k\to\infty.$$

Denote $I_k \equiv I_{k,1} + I_{k,2}$ where $I_{k,1}$ and $I_{k,2}$ are respectively the integrals on the intervals $[a+a_k\cdot g(k)^{-1}, a+\eta_1]$, $[a+\eta_1, b]$. Now by (ii) for each fixed $\eta_1 > 0$

$$|I_{k,2}| \leqq \sqrt{k}\exp[g(k)(h(a+\eta_1)-h(a))]\int_{a+\eta_1}^{b} |\phi_k(u) - \phi_k(a+a_k\cdot g(k)^{-1}+0)|\,du = o(1)\ (k\to\infty).$$

Integration by parts yields

$$I_{k,1} = \sqrt{k}\ \exp[g(k)(h(x)-h(a))\alpha_k(x)]_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} - \sqrt{k}\, g(k)\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} \alpha_k(x)h'(x)\exp[g(k)(h(x)-h(a))]\,dx \equiv I_{k11} + I_{k12}$$

By (ii) and the fact $\alpha_k(a+a_k\cdot g(k)^{-1}) = 0$ we have $I_{k11} = o(1)$, $k\to\infty$. For a given ε_1 let ρ be that existing by (iv); let η_1 be the same as that in the proof of Theorem 1.3 and suppose also $\eta_1 < \rho$.

$$|I_{k12}| \leqq k^{1/2}g(k)\cdot\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} |\alpha_k(x)||\,h'(x)|\exp[g(k)(h(x)-h(a))]\,dx$$

(by (iv))

$$\leqq \varepsilon_1 k^{1/2}\cdot g(k)\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} (x-(a+a_k g(k)^{-1}))\,|h''(\xi(x))(x-a)|\cdot \exp\left[g(k)h''(\xi(x))\frac{(x-a)^2}{2}\right]dx$$

$$\leqq \varepsilon_1 k^{1/2}\cdot g(k)(-h''(a)+\varepsilon)\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} |x-a||x-a-a_k\cdot g(k)^{-1}|\cdot \exp\left[g(k)(h''(a)+\varepsilon)\frac{(x-a)^2}{2}\right]dx$$

(and for $M_1 \equiv -h''(a)+\varepsilon$)

$$\leqq \varepsilon_1 k^{1/2} g(k) M_1 \int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} \exp\left[g(k)(h''(a)+\varepsilon)\frac{(x-a)^2}{2}\right]\cdot$$

$$\cdot\left[(x-a)^2+|x-a||a_k|g(k)^{-1}\right]dx$$

(and by the substitution $u=\sqrt{-(g(k)/2)(h''(a)+\varepsilon)}\,(x-a)$ together with

$$|a_k| \leqq ck^{1/2} \leqq c_1 g(k)^{1/2})$$

$$\leqq M_2\varepsilon_1\int_{-\infty}^{\infty} e^{-u^2}u^2du + M_3\varepsilon_1\int_0^{\infty} ue^{-u^2}du \leqq \varepsilon_1 M_4$$

By letting $\varepsilon_1 \downarrow 0$ we get $I_{k12}=o(1)$ as $k\to\infty$. Q.E.D.

COROLLARY 2.3. *Suppose* (i) $a, b, \eta, \delta, \lambda, h(x), \{a_k\}$ *and* $\{g(k)\}$ *satisfy conditions* (i), (ii), (iii) *and* (iv) *of Corollary* 1.3.

(ii) *For the functions* $\phi_k(u)$ *defined on* $[b, a+a_k\cdot g(k)^{-1}]$ *and for some finite* M *we have*

$$\int_b^{a+a_k\cdot g(k)-1} |\phi_k(x)|\,dx < M \text{ for } k \geqq 1.$$

(iii) $\lim_{k\to\infty}\phi_k(a+a_k\cdot g(k)^{-1}-0)$ *exists and is equal to* A.

(iv) *Denote*

$$\alpha_k(x)=\int_x^{a+a_k\cdot g(k)^{-1}} [\phi_k(u)-\phi_k(a+a_k.g(k)^{-1}-0)]\,dx$$

for $k\geqq 1$ *and* $x\in[b,a+a_k\cdot g(k)^{-1}]$. *Suppose that for each* $\varepsilon_1>0$ *there exists a sufficiently small* $\rho(\varepsilon_1)$, $0<\rho(\varepsilon_1)<(a-b)/2$ *such that for each* $k\geqq 1$ *and all* x *satisfying* $0\leqq(a+a_k\cdot g(k)^{-1})-x\leqq\rho(\varepsilon_1)$ *we have*

$$|\alpha_k(x)| \leqq \varepsilon_1|x-(a+a_k\cdot g(k)^{-1})|.$$

Then

$$\text{(10.3)}\qquad \lim_{k\to\infty}\sqrt{\frac{-kh''(a)}{2\pi}}\,e^{-g(k)h(a)}\int_b^{a+a_k\cdot g(k)^{-1}} \phi_k(x).e^{g(k)h(x)}dx$$

$$=AN(\lambda\sqrt{-h''(a)}).$$

Proof. The same proof as that of Theorem 2.3

THEOREM 3.3. *Suppose that* (i) a, b, p, η *and* δ *are real numbers satisfying* $p<a<b$, $0<\eta<b-a$, $0<\delta<a-p$.

(ii) $h(x) \in C^2(a-\delta \leqq x \leqq a+\eta)$, $h'(a)=0$, $h''(a)<0$, $h(x)$ *is nondecreasing for* $x \in [p,a]$ *and nonincreasing* for $x \in [a,b]$.

(iii) $\{a_k\} \in A^*$.

(iv) $\{g(k)\}$ *satisfies condition* (iv) *of Theorem* 1.3.

(v) *The functions* $\phi_k(u) (k \geqq 1)$ *are defined on* $\rho \leqq x \leqq b$ *and for some finite* M *we have*

$$\int_p^b |\phi_k(u)|\,du < M \qquad \text{for} \qquad k \geqq 1.$$

(vi) *The point* $a + a_k \cdot g(k)^{-1}$ *is a Lebesgue point of* $\phi_k(u)$ *and* $\lim_{k\to\infty}(a + a_k \cdot g(k)_{-6})$ *exists and is equal to* A.

(vii) *Denote*

$$\alpha_k(x) \equiv \int_{a+a_k\cdot g(k)^{-1}}^x [\phi_k(u) - \phi_k(a + a_k \cdot g(a)^{-1})]\,du$$

for $k \geqq 1$ *and* $p \leqq x \leqq b$. *Suppose that for each* $\varepsilon_1 > 0$ *there exists a sufficiently small* $\rho(\varepsilon_1)$, $0 < \rho(\varepsilon_1) < \text{Min}\ ((a-p)/2, (b-a)/2)$ *such that for* $k \geqq 1$ *and all* x *satisfying* $|a + a_k \cdot g(k)^{-1} - x| \leqq \rho(\varepsilon_1)$ *we have* $|\alpha_k(x)| \leqq \varepsilon_1 |x - (a + a_k \cdot g(k)^{-1}|$. *Then*

$$\lim_{k\to\infty} \sqrt{\frac{-kh''(a)}{2\pi}}\, e^{-g(k)h(a)} \int_p^b \phi_k(x) e^{g(k)h(x)}\,dx = A. \tag{11.3}$$

Proof. The same proof as that of Theorem 2.3 because $|a_k| \leqq c \cdot k^{1/2}$.

THEOREM 4.3. *Suppose conditions* (i), (ii), (iii) and (iv) *of Theorem* 2.3 *are satisfied.*

Then

$$\lim_{k\to\infty} -kh''(a)\, e^{-h''(a)\lambda^2/2}\, e^{-g(k)h(a)} \int_{a+a_k\cdot g(k)^{-1}}^b \phi_k(x) \cdot \\ \cdot e^{g(k)h(x)} \cdot (x-a)\,dx = A. \tag{12.3}$$

Proof. The argument used to prove Theorem 1.3 yields also

$$\lim_{k\to\infty} k \int_{a+a_k\cdot g(k)^{-1}}^b (x-a)\exp[g(k)(h(x)-h(a))]\,dx = \frac{-1}{h''(a)}\, e^{h''(a)\lambda^2/2} \tag{13.3}$$

By (13.3), assumption (iii) of Theorem 2.3, and the arguments used in proving Theorem 2.3, it follows that it is enough to show that

$$I_k \equiv k \int_{a+a_k\cdot g(k)^{-1}}^b [\phi_k(x) - \phi_k(a + a_k \cdot g(k)^{-1} + 0)](x-a) \cdot \\ \cdot \exp[g(k)(h(x)-h(a))]\,dx = o(1) \text{ as } k \to \infty.$$

Denote $I_k \equiv I_{k_1} + I_{k_2}$ where I_{k_1} and I_{k_2} are respectively the integrals on $[a + a_k \cdot g(k)^{-1}, a + \eta_1]$ and $[a + \eta_1, b], \eta_1 > 0$. The argument used in the proof of Theorem 2.3 yields, since $|\phi_k(x) - \phi_k(a + a_k \cdot g(k)^{-1} + 0)||x - a|$ is L integrable on $[a + a_k \cdot g(k)^{-1}, b]$, that $I_{k_2} = o(1)$, $k \to \infty$. Integration by parts gives

$$
\begin{aligned}
I_{k_1} = {} & k\alpha_k(x)\exp[g(k)(h(x) - h(a))](x - a)\Big|_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} \\
& - k\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} \alpha_k(x)\exp[g(k)(h(x) - h(a))]\,dx \\
& - kg(k)\int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} \alpha_k(x)h'(x)(x - a)\exp[g(k)(h(x) - h(a))]\,dx \\
= {} & I_{k11} + I_{k12} + I_{k13},
\end{aligned}
$$

where $\alpha_k(x)$ is defined by condition (iv) of Theorem 2.3. Clearly $I_{k11} = o(1)$ as $k \to \infty$. Let $\varepsilon_1 > 0$, $\rho(\varepsilon_1)$, $\varepsilon > 0$, η_1, δ_1 and $\xi(x)$ be the same as in the proof of Theorem 2.3, then for $x \in [a + a_k \cdot g(k)^{-1}, a + \eta_1]$, since $h''(a) + \varepsilon \geqq h''(\xi(x))$,

$$
\begin{aligned}
(14.3)\quad \exp[g(k)(h(x) - h(a))] &= \exp\left[g(k)h''(\xi(x))\,.\,\frac{(x-a)^2}{2}\right] \\
&\leqq \exp\left[g(k)(h''(a) + \varepsilon)\,\frac{(x-a)^2}{2}\right].
\end{aligned}
$$

Let $-L \equiv h''(a) + \varepsilon$. Then by (14.3)

$$
|I_{k12}| \leqq k\varepsilon_1 \int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} (x - a - a_k \cdot g(k)^{-1})\, e^{-g(k)L(x-a)^2/2}\,dx \leqq M_1\varepsilon_1.
$$

Let $h''(a) - \varepsilon \equiv -B$, then, since $-B \leqq h''(\xi) \leqq -L$ we have by (14.3)

$$
|I_{k13}| \leqq kg(k)\varepsilon_1 \int_{a+a_k\cdot g(k)^{-1}}^{a+\eta_1} (x - a - a_k \cdot g(k)^{-1})B(x-a)^2 e^{-g(k)L(x-a)^2/2}\,dx
$$

The substitution $u = \sqrt{g(k)L/2}$ in the last integral together with the inequality $|x - a - a_k \cdot g(k)^{-1}| \leqq |x - a| + |a_k| \cdot g(k)^{-1}$ and the fact $\lim_{k\to\infty} k^{-1} g(k) = 1$ gives

$$
\begin{aligned}
|I_{k13}| &\leqq M_2\varepsilon_1 \int_0^\infty u^3 e^{-u^2}du + M_3\varepsilon_1 |a_k| g(k)^{-1/2} \int_{-\infty}^{\infty} u^2 e^{-u^2}du \\
&\leqq M_4\varepsilon_1.
\end{aligned}
$$

COROLLARY 4.3. *Suppose conditions* (i), (ii), (iii) *and* (iv) *of Corollary* 2.3 *are satisfied.*

Then

$$\lim_{k\to\infty} kh''(a)\,e^{-h''(a)\lambda^2/2}e^{-g(k)h(a)}\int_b^{a+a_k\cdot g(k)^{-1}}\phi_k(x)e^{g(k)h(a)}(x-a)\,dx = A. \tag{15.3}$$

Proof. The same proof as that of Theorem 4.3 .

§4. **Proof of the theorems of** §2. In the proof of Theorem 1.2 we shall use the following result.

LEMMA 1.4. *Suppose that the Laplace transform of* $\phi(u)$ *exists and that* $a_k = o(k)$ *as* $k \to \infty$. *Then*

$$\lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_0^{1-\delta} e^{-kz}z^k\phi\left(z\frac{k}{k+a_k}t\right)dz = 0 \tag{1.4}$$

$$\lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_{1+\delta}^{\infty} e^{-kz}z^k\phi\left(z\frac{k}{k+a_k}t\right)dz = 0 \tag{2.4}$$

Proof. We shall prove (2.4) only. The proof of (1.4) is similar. For real c and $t>0$ let $\alpha(z)\equiv\alpha(z,c,t)\equiv\int_0^z e^{-cu}\phi(ut)\,du$. For any fixed $t>0$ there exist constants $c=c(t)$ and $M=M(t)$ such that

$$|\alpha(z)| \leqq M \equiv M(t) \qquad \text{for } z \geqq 0. \tag{3.4}$$

Define

$$\alpha_k(z) \equiv \int_{1+\delta}^{z} e^{-c(k/(k+a_k))u}\phi\left(u\frac{k}{k+a_k}t\right)du$$

$$= \frac{k+a_k}{k}\int_{(1+\delta)(k/(k+a_k))}^{z(k/(k+a_k))} e^{-cu}\phi(ut)du.$$

Hence, by (3.4)

$$|\alpha_k(z)| \leqq 3M \qquad (k \geqq k_0), \qquad z \geqq 0. \tag{4.4}$$

Integration by parts in (2.4) yields

$$J_k \equiv \frac{k^{k+1}}{k!}\int_{1+\delta}^{\infty} e^{-kz}z^k\phi\left(z\frac{k}{k+a_k}t\right)dz$$

$$= \frac{k^{k+1}}{k!}\alpha_k(u)u^k\exp\left\{-\left(k-c\frac{k}{k+a_k}\right)u\right\}\Big|_{1+\delta}^{\infty}$$

$$-\frac{k^{k+1}}{k!}\int_{1+\delta}^{\infty}\alpha_k(u)\frac{d}{du}\left\{u^k\exp\left[-\left(k-c\frac{k}{k+a_k}\right)u\right]\right\}du$$

The maximum of $u^k\cdot\exp[-(k-c(k/(k+a_k)))u]$ is at $u=(1-(c/(k+a_k)))^{-1}$.

Therefore, by (4.4),

$$|J_k| \leqq 6M \exp\left[(1+\delta)c\frac{k}{k+a_k}\right]\cdot\left\{\frac{1}{k!}e^{-(1+\delta)k}(1+\delta)^k k^{k+1}\right\}$$

for $k \geqq k_0$. The sequence $\{k/(k+a_k)\}$ is bounded, and the argument used in proving Theorem 3a on page 281 of Widder [7] (with $t=1$, $c=0$ there) yields (2.4). Q. E. D.

Proof of Theorem 1.2. Case (i). Let $t>0$ and suppose that $\phi(t\pm 0)$ exist.

$$(5.4)\quad L_{k,t,a_k}[f(x)] = \frac{1}{k!}\left(\frac{k+a_k}{t}\right)^{k+1} e^{-(k+a_k)u/t}\cdot u^k\phi(u)du$$

$$= \frac{k^{k+1}}{k!}\left\{\int_0^{1-\delta}+\int_{1-\delta}^{(k+a_k)/k}+\int_{(k+a_k)/k}^{1+\delta}+\int_{1+\delta}^{\infty}\right\} e^{-kz}z^k\phi\left(z\frac{k}{k+a_k}t\right)dz$$

$$\equiv I_{k1}+I_{k2}+I_{k3}+I_{k4},$$

$\{a_k\}\in A(\lambda)$, and so $a_k=o(k)$, $k\to\infty$. By Lemma 1.4 we have

$$(6.4)\qquad \lim_{k\to\infty} I_{k1} = \lim_{k\to\infty} I_{k4} = 0.$$

In order to find the value of $\lim_{k\to\infty} I_{k3}$ take in Theorem 2.3 $h(z)=\log z - z$, $g(k)=k$, $\phi_k(z)=\phi(z(k/(k+a_k))\cdot t)$ $(k\geqq 1)$, $a=1$ and $b=1+\delta$. It can be verified that the functions $\phi_k(z)$ satisfy conditions (ii), (iii) and(iv) of Theorem 2.3. Hence, by Theorem 2.3 and Stirling's formula, we get

$$(7.4)\qquad \lim_{k\to\infty} I_{k3} = (1-N(\lambda))\phi(t+0).$$

In the same way, using Corollary 2.3 instead of Theorem 2.3, we get

$$(8.4)\qquad \lim_{k\to\infty} I_{k2} = N(\lambda)\phi(t-0)\ .$$

Combining (7.4) and (8.4) we obtain the proof for case (i) of Theorem 1.2.

Proof of Theorem 1.2. Case (v); The proof of this case is the same as that of case (i) but here we use Theorem 3.3 instead of Theorem 2.3.

Proof of Theorem 1.2. Case (*ii*); Here $\{a_k\}\in B^+$. Let $t>0$ and suppose $\phi(t-)$ exists. For $\lambda>0$ and a sequence $\{b_k\}\in A(\lambda)$ we have, for $k\geqq k_0$,

$$1 \geqq \frac{k^{k+1}}{k!}\int_{1-\delta}^{(k+a_k)/k} e^{-kz}z^k dz \geqq \frac{k^{k+1}}{k!}\int_{1-\delta}^{(k+b_k)/k} e^{-kz}z^k dz.$$

Therefore by (8.4), with the function $\phi(t)\equiv 1$, we have

$$1 \geqq \overline{\lim_{k\to\infty}}\int_{1-\delta}^{(k+a_k)/k} e^{-kz}z^k dz \geqq N(\lambda)\,.$$

Now $\lim_{\lambda\uparrow\infty} N(\lambda) = 1$, therefore

$$\lim_{k\to\infty} \int_{1-\delta}^{(k+a_k)/k} e^{-kz} z^k dz = 1 . \tag{9.4}$$

For a given $\varepsilon > 0$ there exists $\delta \equiv \delta(\varepsilon)$, $0 < \delta < 1$ and a constant $k_1 \equiv k_1(\varepsilon_1, \{a_k\})$ such that

$$\left| \phi\left(\frac{k}{k+a_k} tz \right) - \phi(t-) \right| < \varepsilon \text{ for } 1-\delta < z < \frac{k+a_k}{k} \text{ and } k \geqq k_1. \tag{10.4}$$

Let $I_{kj} (1 \leqq j \leqq 4)$ be the same as in (5.4). By (10.4) we have

$$\lim_{k\to\infty} \left| I_{k2} - \phi(t-) \right| \leqq \overline{\lim} \frac{k^{k+1}}{k!} \int_{1-\delta}^{(k+a_k)/k} \left| \phi\left(\frac{ktz}{k+a_k} \right) - \phi(t-) \right| e^{-kz} z^k dz \leqq \varepsilon$$

Letting $\varepsilon \downarrow 0$ we get

$$\lim_{k\to\infty} I_{k2} = \phi(t-). \tag{11.4}$$

The argument used in obtaining (4.4) shows that for fixed $t > 0$ there exist constants $c \equiv c(t)$ and $M_1(t)$ such that

$$|\beta_k(z)| \equiv \left| \int_{(k+a_k)/k}^{z} \exp\left\{ -c \frac{k}{k+a_k} u \right\} \phi\left(\frac{k}{k+a_k} tu \right) du \right| \leqq M \quad \text{for } k \geqq k_0 . \tag{12.4}$$

The only maximum of $z^k \exp\{-k(1-(c/(k+a_k)))z\}$ is at

$$z = (k+a_k)\cdot(k+a_k-c)^{-1}.$$

Also, $\lim_{k\to\infty} a_k = +\infty$, since $\{a_k\} \in B^+$, and so

$$\frac{k+a_k}{k} > \frac{k+a_k}{k+a_k-c} \text{ for } k \geqq k_1 \geqq k_0. \tag{13.4}$$

Integrating by parts the integral defining I_{k3} in (5.4) we get

$$I_{k3} = \frac{k^{k+1}}{k!} e^{-k(1+\delta)} (1+\delta)^k \beta_k(1+\delta) \exp\left\{ \frac{k}{k+a_k} c(1+\delta) \right\}$$
$$- \frac{k^{k+1}}{k!} \int_{(k+a_k)/k}^{1+\delta} \beta_k(z)\, d\left\{ z^k \exp\left[-k\left(1 - \frac{c}{k+a_k}\right) z \right] \right\}.$$

The argument used in proving Lemma 1.4 yields now by (12.4) and (13.4)

$$\lim_{k\to\infty} I_{k3} = 0 . \tag{14.4}$$

By Lemma 1.4 we have $\lim_{k\to\infty} I_{k1} = \lim_{k\to\infty} I_{k2} = 0$. Combining the last result with (11.4) and (14.4) we get the proof of case (ii) of Theorem 1.2.

Proof of Theorem. 1.2. Case (iii). The proof of this case is similar to the proof of case (ii) of Theorem 1.2.

Proof of Theorem 1.2. Case (iv). It is known (and it follows from (9.4) too) that, for $0 < \delta < 1$,

$$\lim_{k\to\infty} \frac{k^{k+1}}{k!} \int_{1-\delta}^{1+\delta} e^{-kz} z^k dz = 1.$$

Since $\phi(u)$ is continuous for $u = t > 0$, therefore for each $\varepsilon > 0$ here is a $\delta \equiv \delta(\varepsilon)$, $0 < \delta < 1$. such that

$$\left| \phi\left(\frac{k}{k+a_k} tz \right) - \phi(t) \right| < \varepsilon \quad \text{for} \quad 1-\delta \leqq z \leqq 1+\delta.$$

Let I_{kj} $(0 \leqq j \leqq 4)$ be defined as in (5.4). Then

$$\overline{\lim_{k\to\infty}} \left| \mathrm{I}_{k2} + \mathrm{I}_{k3} - \phi(t) \right|$$

$$\leqq \overline{\lim_{k\to\infty}} \frac{k^{k+1}}{k!} \int_{1-\delta}^{1+\delta} e^{-kz} z^k \left| \phi\left(\frac{k}{k+a_k} \cdot tz \right) - \phi(t) \right| dz \leqq \varepsilon.$$

Letting $\varepsilon \downarrow 0$ we get $\lim_{k\to\infty} \{I_{k2} + I_{k3}\} = \phi(t)$. By Lemma 1.4 we have $\lim_{k\to\infty} I_{k1} = \lim_{k\to\infty} I_{k4} = 0$. This completes the proof of the case (iv) of Theorem 1.2.

Proof of Theorem 2.2. By supposition $\alpha(0) = 0$. It is known (Widder [7]) that if $f(x)$ is the Laplace-Stieltjes transform of $\alpha(t)$ then $f(x)/x$ is the Laplace transform of $\alpha(t)$. If $f(x)$ is the Laplace-Stieltjes transform of $\alpha(t)$ them (See [7], p. 294),

$$\frac{(-1)^k}{k!} \left[\frac{f(x)}{x} \right]^{(k)} x^{k+1} = f(\infty) + (-1)^{k+1} \int_x^{\infty} \frac{u^k}{k!} f^{(k+1)}(u) du.$$

Substituting $x = (k + a_k)/t$ we get by (7.2) and (8.2)

$$L_{k,t,a_k}\left[\frac{f(x)}{x} \right] = S_{k,t,a_k}[f(x)].$$

The proof follows now from Theorem 1.2 since $\alpha(0) = 0$, $f(x)/x$ is the Laplace transform of $\alpha(t)$ and for each $t > 0$ $\alpha(t \pm)$ exists, therefore $\alpha(t \pm 0)$ exists and respectively $\alpha(t \pm 0) = \alpha(t \pm)$. Q.E.D.

Proof of Theorem 3.2. The argument used by Widder (see [7] p. 291) yields for $0 < r < t$

$$(15.4)\quad J_{r,t,k} \equiv \int_r^t L_{k,u,a_k}[f(s)]\,du$$

$$= \frac{k+a_k}{k!}\left(\frac{k+a_k}{t}\right)^k \int_0^\infty e^{-(k+a_k)y/t}\, y^{k-1}\alpha(y)dy$$

$$- \frac{k+a_k}{k!}\left(\frac{k+a_k}{r}\right)^k \int_0^\infty e^{-(k+a_k)y/r} y^{k-1}\alpha(y)dy \equiv J_{k,t} - J_{k,r}$$

Now, Theorem 1.2 yields for a fixed $t > 0$ (because if $\{a_k\}$ belongs to any one of the classes $A(\lambda), B^{\pm}$ and B then $\{a_k+1\}$ belongs to the same class and $(k+a_k)/k$ $(k+a_k)/k \to 1$ as $k \to \infty$)

$$(16.4)\qquad \lim_{k\to\infty} J_{k,t} = \begin{cases} N(\lambda)\alpha(t-) + (1-N(\lambda))\alpha(t+) & \text{if } \{a_k\} \in A(\lambda) \\ \alpha(t-) & \text{if } \{a_k\} \in B^+ \\ \alpha(t+) & \text{if } \{a_k\} \in B^- \\ \alpha(t) & \text{if } \alpha(t+) = \alpha(t-) \text{ and } \{a_k\} \in B \end{cases}$$

$$(17.4)\qquad J_{k\,r} - \frac{k+a_k}{k}\alpha(0+) = \frac{k+a_k}{k!}\left(\frac{k+a_k}{r}\right)^k \cdot \int_0^\infty e^{-(k+a_k)y/r}\,\cdot$$
$$\cdot\, y^{k-1}[\alpha(y) - \alpha(0+)]\,dy \qquad k \to \infty.$$

Now the argument used in [7] at the foot of page 291 and the top of 292 yields

$$(18.4)\qquad \lim_{r\downarrow 0} J_{k,r} = \alpha(0+)\cdot\frac{k+a_k}{k}.$$

If $\{a_k\}$ belongs to any one of the classes $A(\lambda)$, $B^{\pm}$ and B then $(k+a_k)/k \to 1$. Hence by (18.4)

$$(19.4)\qquad \lim_{k\to\infty}\lim_{r\downarrow 0} J_{k,r} = \alpha(0+)$$

The proof follows now by combining (15.4), (16.4) and (19.4).

Proof of Theorem 4.2. For a fixed $t > 0$ and $\{a_k\} \in A(\lambda)$, denote

$$I_k \equiv e^{\lambda^2/2}\frac{\sqrt{2\pi k}}{k!}(-1)^{k+1}\left\{\left(\frac{k+a_k}{t}\right)^k \frac{k}{t} f^{(k)}\left(\frac{k+a_k}{t}\right) + \frac{1}{t}\left(\frac{k+a_k}{t}\right)^{k+1} f^{(k+1)}\left(\frac{k+a_k}{t}\right)\right\}.$$

By Theorem 1.2 Case (i) we get

$$\lim_{k\to\infty}\frac{\theta_k\sqrt{k}}{k!}\left(\frac{k+a_k}{t}\right)^k f^{(k)}\left(\frac{k+a_k}{t}\right) = 0.$$

Comparing the definition of I_k and the result of our theorem we see that in order to prove our theorem it is enough to show that

$$\lim_{k\to\infty} I_k = \phi(t+0) - \phi(t-0). \tag{20.4}$$

We have

$$I_k = \phi^{\lambda^2/2}\frac{\sqrt{2\pi k}}{k!}\left(\frac{k+a_k}{t}\right)^k \int_0^\infty \left\{-\frac{k}{t}u^k + \frac{1}{t}\left(\frac{k+a_k}{t}\right)u^{k+1}\right\}\cdot \tag{21.4}$$

$$\cdot e^{-(k+a_k)u/t}\phi(u)du = e^{\lambda^2/2}\,\frac{\sqrt{2\pi k}}{k!}\cdot\frac{k^{k+2}}{k+a_k}\cdot$$

$$\cdot\left\{\int_0^{1-\delta} + \int_{1-\delta}^{(k+a_k)/k} + \int_{(k+a_k)/k}^{1+\delta} + \int_{1+\delta}^{\infty}\right\} e^{-kz}z^k(z-1)\phi\left(\frac{tkz}{k+a_k}\right)dz$$

$$\equiv I_{k,1} + I_{k,2} + I_{k,3} + I_{k,4}.$$

The arguments used in proving Lemma 1.4 yield here

$$\lim_{k\to\infty} I_{k1} = \lim_{k\to\infty} I_{k4} = 0. \tag{22.4}$$

In order to estimate $I_{k,3}$ we substitute in Theorem 4.3 $a = 1$, $b = 1+\delta$, $h(z) = -z + \log z$, $g(k) = k$ and $\phi_k(z) = \phi(tkz/(k+a_k))$ and get by Theorem 4.3

$$\lim_{k\to\infty} I_{k3} = \phi(t+0). \tag{23.4}$$

In the same way, but using Corollary 4.3 instead of Theorem 4.3 we get

$$\lim_{k\to\infty} I_{k2} = -\phi(t-0). \tag{24.4}$$

Combining (21.4), (22.4), (23.4) and (24.4) we get (20.4) and this completes the proof of our theorem.

Proof of Theorem 5.2. For a fixed $t > 0$ and $\{a_k\} \in A(\lambda)$ denote

$$I_k \equiv e^{\lambda^2/2}\sqrt{2\pi k}\left(\frac{k+a_k}{t}\right)^k\frac{(-1)^k}{k!}f^{(k)}\left(\frac{k+a_k}{t}\right)$$

$$= e^{\lambda^2/2}\,\frac{\sqrt{2\pi k}}{k!}\left(\frac{k+a_k}{t}\right)^k\int_0^\infty e^{-(k+a_k)u/t}u^k d\alpha(t)$$

$$= e^{\lambda^2/2}\,\frac{\sqrt{2\pi k}}{k!}\left(\frac{k+a_k}{t}\right)^k\int_0^\infty e^{-(k+a_k)u/t}\left(ku^{k-1} - \left(\frac{k+a_k}{t}\right)u^k\right)\alpha(u)du$$

$$= e^{\lambda^2/2}\,\frac{\sqrt{\pi 2k}}{k!}\,k^{k+1}\int_0^\infty e^{-kz}z^{k-1}(1-z)\,\alpha\left(\frac{kzt}{k+a_k}\right)dz.$$

The argument used in proving Theorem 4.2, but taking here $(1/z)\alpha(kzt/(k+a_k))$ instead of $\phi(kzt/(k+a_k))$ there completes the proof.

Added in proof. Theorem 2 of the paper by L. C. Hsu "Generalized Stieltjes-Post Inversion formula for Integral transforms involving a parameter," *Amer. J. Math.*, **73** (1951), 199–210, is Pollard's Theorem 1.1 of [4]. As we have shown in §2 there is an incorrect step in Pollard's proof. There is also an incorrect step in Hsu's proof of his Lemma 2 which is used in proving his Theorem 2. Hsu proves on page 204 that if a sequence $\{x_n\}$ of Lebesgue's points of a function $f(x)$ converges to a Lebesgue's point x of the same function then $\lim_{n\to\infty} f(x_n) = f(x)$ The following example shows that this is not true. Define $f(x)$ in $[-1,\frac{1}{4}]$ by $f(x) = 0$ for $-1 \leqq x \leqq 0$, $f(x) = 1$ for $4^{-n} - 4^{-n}/2n \leqq x \leqq 4^{-n}$ $(n \geqq 1)$ and $f(x) = 0$ in all remaining points of $[-1,\frac{1}{4}]$. The points $x_n = 4^{-n} - 4^{-(n+1)}/n$ and $x = 0$ for $f(x)_n = 1, f(0) = 0$ are Lebesgue's points of $f(x)$. But $x_n \to 0$ and $f(x_n) \to 1 \neq 0 = f(0)$.

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The Hebrew University of
Jerusalem